Trading Networks with General Preferences

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Abstract

We establish the lattice theorem, rural hospitals theorem, and a group-incentive-compatibility result for terminal buyers (sellers) with unit demand, in a general bilateral trading network without making the assumption of quasi-linear utility in transfers. *JEL-classification:* C78, D47
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1 Introduction

Substitutability is a key concept for the analysis of markets with indivisibilities that feature prominently in the field of market design (see e.g. Milgrom, 2017 who highlights the role of substitutability in market design). Under substitutability, equilibria in markets with indivisibilities exist, natural tâtonnement dynamics converge to equilibrium, (one-sided) incentive compatibility results for mechanisms that implement equilibria hold, competitive equilibria have cooperative foundations and one can obtain important structural results, such as the lattice structure of competitive equilibrium prices. The importance of substitutability in markets with indivisibilities has first been highlighted by Kelso and Crawford (1982). They showed that in a two-sided labor market matching model, under the assumption of gross substitutability, a natural ascending auction converges to an approximate equilibrium which becomes a competitive equilibrium as price increments become

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smaller and smaller. In the last three decades, the work of Kelso and Crawford (1982) has been vastly generalized beyond two-sided markets and additional results have been obtained. Recently, the results have been extended to trading networks, which allow to model complex supply chains in an industry where firms are engaged in upstream as well as downstream contracts. Hatfield et al. (2013) show that under the assumption of full substitutability,[1] which requires that firms see upstream (downstream) contracts as substitutes to each other and upstream (downstream) contracts as complements to downstream (upstream) contracts, the set of competitive equilibria is non-empty, has a lattice structure, and is essentially equivalent to the core and the set of (pairwise) stable allocations.

In Hatfield et al. (2013) as well as many precursors, quasi-linearity in transfers is assumed to simplify the analysis. However, recent studies (Fleiner et al., 2018; Dupuy et al., 2017; Galichon et al., 2017; Hatfield et al., 2018a) have gone beyond the quasi-linear benchmark to capture important frictions (such as taxation) or wealth effects that are present in real-world markets. It was unknown so far, whether the structural properties of the set of competitive equilibria carry over to the case of non-quasi-linear utility. In contrast to this, in models with discrete transfers or no transfers (Fleiner, 2003; Hatfield and Milgrom, 2005; Ostrovsky, 2008; Fleiner et al., 2016), only the assumption of strict preferences, (full) substitutability and the law of aggregate demand (supply) are sufficient for the set of (trail-)stable outcomes, which is the (cooperative) solution concept most often used in this context,[2] to exhibit the same structural properties as competitive equilibria for models with transfers. In particular, in the discrete world with strict preferences, severe income effects can be handled as long as substitutability and the law of aggregate demand (supply) are not violated. Since competitive equilibria in market with continuous transfers have cooperative foundations even without quasi-linearity, as recently demonstrated by Fleiner et al. (2018), it seems plausible that the set of competitive equilibria, and hence by the equivalence result of Fleiner et al. (2018), (trail)-stable allocations, exhibit the same structural properties as for quasi-linear preferences. The purpose of this paper is to show that this is indeed the case and hence full substitutability and the laws of aggregate demand/supply alone are driving all of the results.

We show that for a fully general model of trading networks with transfers, only assuming continuity and monotonicity in transfers, the set of competitive equilibria has a lattice structure, provided that full substitutability and the laws of aggregate demand and supply hold. Moreover, under our assumptions, we show

[1]Full Substitutability has first been introduced by Ostrovsky (2008) for a discrete model of trading chains and generalizes a condition first introduced by Sun and Yang (2006) in the context of exchange economies with indivisibilities).

[2]Stability has appealing normative properties, as well as empirical support, see e.g. Roth (1984a, 1991).

that a version of the rural hospitals theorem holds. If we make the additional assumption of bounded willingness to pay (as introduced by Fleiner et al., 2018) we obtain an existence result of extreme equilibria: There exists an equilibrium that is most preferred by terminal sellers and an equilibrium that is most preferred by terminal buyers among all equilibria. The latter result allows us to obtain a group-incentive compatibility result. We show that a mechanism that selects terminal buyer optimal equilibria is group-strategy-proof for terminal buyers on the domain of unit demand utility functions and similarly a mechanism that selects terminal seller optimal equilibria is group-strategy-proof for terminal sellers on the domain of unit supply utility functions. Hence we obtain continuous analogs for the full set of results that have been previously obtained for discrete markets. This highlights the similarity between the discrete and continuous models and the fact that quasi-linearity is not essential for any of the canonical results in the literature.

1.1 Related Literature

The literature on trading networks has its origins in the literature on matching markets with transfers. In a seminal paper, Kelso and Crawford (1982) show that, under the assumption of gross substitutability, competitive equilibria with personalized prices exist and are equivalent to core allocations in a many-to-one labor market matching model. The construction is by an approximation argument where the existence in the continuum is obtained from the existence of an equilibrium in a discrete markets with smaller and smaller price increments. Subsequent work has further studied the question of existence of equilibria in the context of exchange economies with indivisibilities. See for example Gul and Stacchetti (1999) and the recent contribution of Baldwin and Klemperer (2016). Different versions of a (group-)strategy-proofness result for a many-to-one matching model with continuous transfers have been established by Hatfield et al. (2014); Schlegel (2016); Jagadeesan et al. (2018).

Trading networks with bilateral contracts and continuous transfers have been introduced by Hatfield et al. (2013). Under the assumption of quasi-linear utility and full substitutability they establish many results that we generalize to the case of general utility functions. The notion of full substitutability has been studied in detail by Hatfield et al. (2018b) who show the equivalence of various different definitions of full substitutability. The work of Hatfield et al. (2013) builds on the work of Ostrovsky (2008) on trading networks with discrete contracts that generalizes matching models with contracts (Hatfield and Milgrom, 2005; Fleiner, 2003; Roth, 1984b) beyond two-sided markets. The matching model with contracts in turn originates in the discrete version of the model of Kelso and Crawford (1982). Hatfield and Kominers (2012) and Fleiner et al. (2016) provide additional results for the discrete trading networks model, which in many ways are parallel to the results we obtain in the continuous model.

All the above mentioned work for continuous models make the assumption of quasi-linear utility for at least one side of the market.[3] There are two papers that deal with general, not necessarily quasi-linear, utility functions and are particularly close to our work:

In a classical paper, Demange and Gale (1985) establish several structural results about the core (or equivalently the set of competitive equilibria) for a one-to-one matching model with continuous transfers. In particular, they show that the core has a lattice structure and an agent that is unmatched in one core allocation receives his reservation utility in each core allocation (the result is often called the rural hospital theorem in the literature on discrete matching markets). Moreover, they show that the mechanism that selects an extreme point of the bounded lattice is strategy-proof for one side of the market. Importantly, these results are established without assuming quasi-linearity in transfers. They only require that utility is increasing, unbounded and continuous in transfers. We generalize this work to trading networks with bilateral contracts.

In recent work Fleiner et al. (2018) study trading networks with general preferences. Their work is in many regards complementary to our work and all major results of Hatfield et al. (2013) are generalized to non-quasi-linear preferences, either in our work or by Fleiner et al. (2018). The authors establish the existence of a competitive equilibrium under the assumption of Full Substitutability and mild regularity conditions, using the approximation approach pioneered by Kelso and Crawford (1982). Moreover, they provide conditions under which competitive equilibria correspond to (trail-)stable allocations. We derive our results for competitive equilibria. However, by the equivalence result of Fleiner et al. (2018) analogous results also would hold for trail-stable allocations.

2 Model

We consider a finite set of **firms** F and a finite set of **trades** Ω. Each trade $\omega \in \Omega$ is associated with a buyer $b(\omega) \in F$ and a seller $s(\omega) \in F$ with $b(\omega) \neq s(\omega)$. For a set of trades $\Psi \subseteq \Omega$ and firm $f \in F$ we define the set of **downstream trades** for f by $\Psi_{f\to} := \{\omega \in \Psi : s(\omega) = f\}$ and the set of **upstream trades** by $\Psi_{\to f} := \{\omega \in \Omega : b(x) = f\}$. Moreover, we let $\Psi_f := \Psi_{f\to} \cup \Psi_{\to f}$. A firm $f \in F$ such that $\Omega_{f\to} = \emptyset$ is called a **terminal buyer** and a firm such that $\Omega_{\to f} = \emptyset$ is called a **terminal buyer**. Note that terminal buyers and/or terminal buyers do

[3]Note however that the existence proof of Kelso and Crawford (1982) is actually more general and also applies to non-quasi-linear preferences, provided they are continuous, monotonic and unbounded in transfers for each bundle.

not need to exist. A **contract** is a pair $(\omega, p_\omega) \in \Omega \times \mathbb{R}$, where p_ω is the price attached to the trade ω.

An **allocation** is a pair (Ψ, p) consisting of a set of trades $\Psi \subseteq \Omega$ and a price vector $p \in \mathbb{R}^\Psi$. We denote the set of allocations by $\mathcal{A}$ and we let $\mathcal{A}_f := \{(\Psi_f, (p_\omega)_{\omega\in\Psi_f}) : (\Psi, p) \in \mathcal{A}\}$. An **arrangement** is a pair $[\Psi, p] \in 2^\Omega \times \mathbb{R}^\Omega$. Thus in contrast to an allocation the price vector also contains prices for unrealized trades.

2.1 Utility functions

Each firm has a utility function $u^f : \mathcal{A}_f \to \mathbb{R}\cup\{-\infty\}$. As Hatfield et al. (2013), we allow the utility function to take on a value of $-\infty$ to model technical constraints faced by a firm. We require that $u^f(\Psi, p) = -\infty$ implies $u^f(\Psi, p') = -\infty$ for each $p' \in \mathbb{R}^\Psi$ and that $u^f(\emptyset) > -\infty$. For notational convenience we extend u^f to $2^\Omega \times \mathbb{R}^\Omega$ by defining for $\Psi \subseteq \Omega$ and $p \in \mathbb{R}^\Omega$, the utility $u^f(\Psi, p) := u^f(\Psi_f, (p_\omega)_{\omega\in\Psi_f})$. We make the following assumptions on utility functions:

- **Continuity**: For each $\Psi \subseteq \Omega_f$ with $u^f(\Psi, \cdot) > -\infty$ the functions $u^f(\Psi, \cdot)$ is continuous on $\mathbb{R}^\Psi$.
- **Monotonicity**: For $\Psi \subseteq \Omega_f$ with $u^f(\Psi, \cdot) > -\infty$ and $p, p' \in \mathbb{R}^\Psi$ with $p' \neq p$:
  1. If $p'_\omega = p_\omega$ for $\omega \in \Psi_{f\to}$ and $p_\omega \leq p'_\omega$ for $\omega \in \Psi_{\to f}$, then $u^f(\Psi, p) > u^f(\Psi, p')$.
  2. If $p'_\omega = p_\omega$ for $\omega \in \Psi_{\to f}$ and $p_\omega \geq p'_\omega$ for $\omega \in \Psi_{f\to}$, then $u^f(\Psi, p) > u^f(\Psi, p')$.

Thus utility is continuous in prices and firms prefer higher sell prices strictly to lower sell prices and lower buy prices strictly to higher buy prices.

A special case, studied in previous work (Hatfield et al., 2013) is the case where transfers enter utility linearly. In this case there is valuation function $v^f : 2^{\Omega_f} \to \mathbb{R} \cup \{-\infty\}$ such that

$$u^f(\Psi, p) = v^f(\Psi) + \sum_{\omega\in\Psi_{f\to}} p_\omega - \sum_{\omega\in\Psi_{\to f}} p_\omega.$$

A utility functions induces a **demand correspondence** $D^f : \mathbb{R}^\Omega \rightrightarrows 2^{\Omega_f}$ by:

$$D^f(p) := \text{argmax}_{\Psi\subseteq\Omega_f} u^f(\Psi, p).$$

It is a straightforward consequence of the continuity of the utility function that the demand correspondence satisfies the following continuity property:

Upper Hemi-Continuity: Let $\|\cdot\|$ be the Euclidean norm.[4] The demand corre-

[4] As usual, we could replace the Euclidean norm by any norm on $\mathbb{R}^{\Omega_f}$.

spondence D^f is upper-hemicontinuous, if for each $p \in \mathbb{R}^{\Omega_f}$ there is an $\epsilon > 0$ such for any $q \in \mathbb{R}^{\Omega_f}$ with $\|p - q\| < \epsilon$, we have $D^f(q) \subseteq D^f(p)$.

Lemma 1. *For a continuous utility function u^f the induced demand D^f is upper hemi-continuous.*

For completeness, the appendix contains a proof of the lemma.

2.2 Full substitutability and the laws of aggregate demand and supply

We now introduce several properties (same side substitutability, cross-side complementarity and their combination called full substitutability) of demand functions that have been well-studied in the literature. We use the "demand language" (Hatfield et al., 2018b) definitions of the properties.

The following two properties have been originally introduced by Ostrovsky (2008) for a discrete model of trading networks with contracts.

Same-Side Substitutability (SSS): For $p, p' \in \mathbb{R}^\Omega$ and each $\Psi' \in D^f(p')$ there exists a $\Psi \in D^f(p)$ such that if $p_\omega = p'_\omega$ for $\omega \in \Omega_{f\to}$ and $p_\omega \leq p'_\omega$ for $\omega \in \Omega_{\to f}$, then

$$\{\omega \in \Psi_{\to f} : p_\omega = p'_\omega\} \subseteq \Psi'_{\to f},$$

and if $p_\omega = p'_\omega$ for $\omega \in \Omega_{\to f}$ and $p_\omega \geq p'_\omega$ for $\omega \in \Omega_{f\to}$, then

$$\{\omega \in \Psi_{f\to} : p_\omega = p'_\omega\} \subseteq \Psi'_{f\to}.$$

Cross-Side Complementarity (CSC): For $p, p' \in \mathbb{R}^\Omega$ and each $\Psi' \in D^f(p')$ there exists a $\Psi \in D^f(p)$ such if $p_\omega = p'_\omega$ for $\omega \in \Omega_{f\to}$ and $p_\omega \leq p'_\omega$ for $\omega \in \Omega_{\to f}$, then

$$\Psi'_{f\to} \subseteq \Psi_{f\to},$$

and if $p_\omega = p'_\omega$ for $\omega \in \Omega_{\to f}$ and $p_\omega \geq p'_\omega$ for $\omega \in \Omega_{f\to}$, then

$$\Psi'_{\to f} \subseteq \Psi_{\to f}.$$

The combination of the two properties is called full substitutability.

Full Substitutability (FS): The demand of firm f satisfies full substitutability if it satisfies Same-Side Substitutability and Cross-Side Complementarity.

Remark 1. As observed, by Hatfield et al. (2018b) in their Appendix A, there is an "expansion version" and a "contraction version" of full substitutability that differ with regard to how the conditions are defined at price vectors where multiple bundles of trades are optimal. Our definition uses the expansion version of the properties. Hatfield et al. (2018b) show that under the assumption of quasi-linearity, the expansion and contraction version of full substitutability are equivalent. Without quasi-linearity, however, the equivalence no longer holds. We discuss in Appendix B how several different versions of the properties are logically related.

Under quasi-linear utility functions, a weaker version of full substitutability in which the property is only imposed at price vectors where the demand is single-valued is equivalent to our version of full substitutability (Hatfield et al., 2018b). We call this property weak full substitutability.[5]

Weak Full Substitutability: For $p, p' \in \mathbb{R}^{\Omega}$ such that $D^f(p) = \{\Psi\}$ and $D^f(p') = \{\Psi'\}$, if $p_\omega = p'_\omega$ for $\omega \in \Omega_{f\to}$ and $p_\omega \leq p'_\omega$ for $\omega \in \Omega_{\to f}$, then

$$\{\omega \in \Psi_{\to f} : p_\omega = p'_\omega\} \subseteq \Psi'_{\to f} \text{ and } \Psi'_{f\to} \subseteq \Psi_{f\to},$$

and if $p_\omega = p'_\omega$ for $\omega \in \Omega_{\to f}$ and $p_\omega \geq p'_\omega$ for $\omega \in \Omega_{f\to}$, then

$$\{\omega \in \Psi_{f\to} : p_\omega = p'_\omega\} \subseteq \Psi'_{f\to} \text{ and } \Psi'_{\to f} \subseteq \Psi_{\to f}.$$

The following counter-example shows that weak full substitutability is strictly weaker than full substitutability for general utility functions. In Section 3, we will use the counter-example to show that under weak full substitutability, the results in our paper do not necessarily hold. See Figure 1 for a geometric representation of the demand in the example.

Example 1. Consider four trades $\Omega = \{\alpha^1, \alpha^2, \beta^1, \beta^2\}$ with $f = b(\alpha^1) = b(\alpha^2) = s(\beta^1) = s(\beta^2)$. We let $u^f(\emptyset) = 0$, $u^f(\{\alpha^i, \beta^j\}, p_{\alpha^i}, p_{\beta^j}) = 2 - p_{\alpha^i} + p_{\beta^j}$ for $i, j, = 1, 2$, and

$$u^f(\{\alpha^1, \alpha^2, \beta^1, \beta^2\}, p) = 4 - \exp\left(\frac{p_{\alpha^1} + p_{\alpha^2}}{2} - 1\right) - \exp\left(1 - \frac{p_{\beta^1} + p_{\beta^2}}{2}\right).$$

We let $u^f(\Psi, p) = -\infty$ for each other $\Psi \subseteq \Omega$. Observe that

$$D^f(1,1,1,1) = \{\{\alpha^1, \beta^1\}, \{\alpha^1, \beta^2\}, \{\alpha^2, \beta^1\}, \{\alpha^2, \beta^2\}, \{\alpha^1, \alpha^2, \beta^1, \beta^2\}\}$$

but

$$D^f(0,1,1,1) = \{\{\alpha^1, \beta^1\}, \{\alpha^1, \beta^2\}\}.$$

[5]Fleiner et al. (2018) call this property "full substitutability". Thus they reserve the term full substitutability for the weaker notion and all of their results hold for the weaker notion of full substitutability. Our use of the term full substitutability is consistent with the use of the term in Hatfield et al. (2018a). They establish the equivalence of chain stability and stability in trading networks for general utility functions under the assumption of the stronger version of full substitutability that we also use.

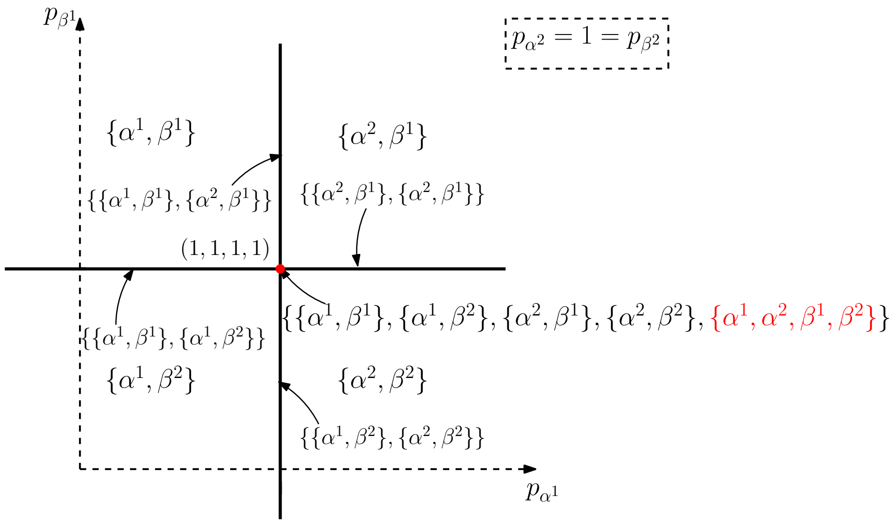

Figure 1: The demand in price space for $p_{\alpha^2} = 1 = p_{\beta^2}$.

As $\{\alpha^1, \alpha^2, \beta^1, \beta^2\} \in D^f(1, 1, 1, 1)$, Full Substitutability would require that there is a $\Psi \in D^f(0, 1, 1, 1)$ with $\{\beta^1, \beta^2\} \subseteq \Psi$. Hence Full Substitutability is not satisfied. As the demand at $(0, 1, 1, 1)$ and $(1, 1, 1, 1)$ is multi-valued, Weak Full Substitutability does not impose any structure here. More generally, note that if we replace u^f by the quasi-linear utility functions $\tilde{u}^f$ such that $\tilde{u}^f(\{\alpha^1, \alpha^2, \beta^1, \beta^2\}, p) = -\infty$ for all $p \in \mathbb{R}^\Omega$ and u^f remains otherwise unchanged, only the demand at prices $(1, 1, 1, 1)$ changes. One readily checks that $\tilde{u}^f$ satisfies Full Substitutability. Hence u^f satsfies Weak Full Substitutability. □

The other two important properties that we consider throughout the paper are monotonicity properties called the Law of Aggregate Demand respectively the Law of Aggregate Supply.[6] Under quasi-linear utility functions, the two properties are implied by (weak) full substitutability. However, in general they are independent of full substitutability.

Law of Aggregate Demand (LAD): For $p, p' \in \mathbb{R}^\Omega$ and each $\Psi' \in D^f(p')$ there exists a $\Psi \in D^f(p)$ such that if $p_\omega = p'_\omega$ for $\omega \in \Omega_{f\to}$ and $p_\omega \leq p'_\omega$ for $\omega \in \Omega_{\to f}$, then

$$|\Psi_{\to f}| - |\Psi_{f\to}| \geq |\Psi'_{\to f}| - |\Psi'_{f\to}|.$$

Law of Aggregate Supply (LAS): For $p, p' \in \mathbb{R}^\Omega$ and each $\Psi' \in D^f(p')$ there exists a $\Psi \in D^f(p)$ such that if $p_\omega = p'_\omega$ for $\omega \in \Omega_{\to f}$ and $p_\omega \geq p'_\omega$ for $\omega \in \Omega_{f\to}$,

[6]The definitions are the demand-language versions of the (choice-language) definitions of Hatfield et al. (2018b). See Definition 10 in their paper.

then

$$|\Psi_{f\to}| - |\Psi_{\to f}| \geq |\Psi'_{f\to}| - |\Psi'_{\to f}|.$$

The combination of full substitutability and the laws of aggregate demand and supply imply an invariance property of the demand that will be crucial for many of our results. It states that if a bundle of trades is demanded by a firm f at a price vector, then this bundle is also demanded by f, if all the trades in the bundle in which f is a seller become more expensive, all trades not in the bundle in which f is a seller become cheaper, all trades in the bundle in which f is a buyer become cheaper, and all trades not in the bundle in which f is a seller become more expensive.

Lemma 2. *Let D^f satisfy FS, LAD and LAS. Let $p, p' \in \mathbb{R}^\Omega$ and $\Psi' \in D^f(p')$. If $p_\omega \leq p'_\omega$ for $\omega \in \Omega_{f\to} \setminus \Psi'_{f\to}$, $p_\omega \geq p'_\omega$ for $\omega \in \Psi'_{f\to}$, $p_\omega \geq p'_\omega$ for $\omega \in \Omega_{\to f} \setminus \Psi'_{\to f}$ and $p_\omega \leq p'_\omega$ for $\omega \in \Psi'_{\to f}$, then $\Psi' \in D^f(p)$. Moreover, if all of the inequalities are strict, then $D^f(p) = \{\Psi'\}$.*

3 Results

3.1 The Lattice Theorem and the Rural Hospitals Theorem

We now introduce the solution concept of a competitive equilbrium (with trade specific prices) and establish that equilibrium prices form a lattice and that (modulo indiferences) for each firm the difference between the number of signed upstream and downstream contracts is the same in each equilibrium. These results extend results established by Hatfield et al. (2013) for the case of quasi-linear utility functions.

In the following, a **competitive equilibrium** for utility profile $u = (u^f)_{f\in F}$ is an arrangement $[\Psi, p] \in 2^\Omega \times \mathbb{R}^\Omega$ such that for each $f \in F$ and the demand D^f induced by u^f we have $\Psi_f \in D^f(p)$. We call $(\Psi, (p_\omega)_{\omega\in\Psi})$ the **equilibrium allocation** induced by $[\Psi, p]$. We denote the set of equilibrium price vectors for u by $\mathcal{E}(u)$ and define for each price vector $p \in \mathbb{R}^\Omega$ the (possibly empty) set $\mathcal{E}(u, p) := \{\Psi \subseteq \Omega : \Psi_f \in D^f(p) \text{ for each } f \in F\}$ of sets of trades that support p as a competitive equilibrium under u.

Theorem 1. *Let u be a utility profile such that for each firm the induced demand satisfies Full Substitutability and the Laws of Aggregate Demand and Supply.*

1. **Lattice Theorem***: Let $p, p' \in \mathcal{E}(u)$ be competitive equilibrium prices. Then the price vectors $\bar{p}, \underline{p} \in \mathbb{R}^\Omega$ defined by*

$$\bar{p}_\omega := \max\{p_\omega, p'_\omega\}, \quad \underline{p}_\omega := \min\{p_\omega, p'_\omega\}$$

are competitive equilibrium prices.

2. **Rural Hospitals Theorem**: *Let $p, p' \in \mathcal{E}(u)$ be competitive equilibrium prices. For each $\Psi \in \mathcal{E}(u,p)$ there exists a $\Psi' \in \mathcal{E}(u,p')$ such that for each $f \in F$ we have $|\Psi_{\to f}| - |\Psi_{f\to}| = |\Psi'_{\to f}| - |\Psi'_{f\to}|$.*

The theorem fails to hold if we replace full substitutability by weak full substitutability.

Example 1 (cont.). Consider the set of trades $\Omega = \{\alpha^1, \alpha^2, \beta^1, \beta^2\}$ and firm f with the utility function u^f as defined in Example 1. The induced demand D^f satisfies weak full substitutability as previously shown. Moreover, for each $p \in \mathbb{R}^\Omega$ and $\Psi \in D^f(p)$ we have $|\Psi_{f\to}| = |\Psi_{\to f}|$. Thus D^f satisfies the Laws of Aggregate Demand and Supply. Consider four additional firms s^1, s^2, b^1, b^2 with $s^1 = s(\alpha^1), s^2 = s(\alpha^2), b^1 = b(\beta^1)$ and $b^2 = b(\beta^2)$. Define utility functions for the additional firms as follows: For $i = 1,2$ define $u^{s^i}(\{\alpha^i\}, p_{\alpha^i}) = p_{\alpha^i}$ and $u^{b^i}(\{\beta^i\}, p_{\beta^i}) = 2 - p_{\beta^i}$ and $u^{s^i}(\emptyset) = u^{b^i}(\emptyset) = 0$. Observe that the equilibria for u are $[\Omega, (1,1,1,1)]$ and $[\{\alpha^i, \beta^j\}, (0,0,2,2)]$ for $i,j = 1,2$. In particular, the vector $(1,1,2,2)$ is not an equilibrium price vector, since $D^{s^1}(1,1,2,2) = \{\{\alpha^1\}\}$ and $D^{s^2}(1,1,2,2) = \{\{\alpha^2\}\}$ but $D^f(1,1,2,2) = \{\{\alpha^1,\beta^1\}, \{\alpha^1,\beta^2\}, \{\alpha^2,\beta^1\}, \{\alpha^2,\beta^2\}\}$. Similarly, $(0,0,1,1)$ is not an equilibrium price vector. □

3.2 Compactness and extremal points of the lattice

So far we have not considered the question of existence of competitive equilibria and, in principle, the lattice in Theorem 1 could be empty. However, as shown by Fleiner et al. (2018), under the assumption of (weak) full substitutability and a very mild regularity condition called bounded compensating variations there always exists a competitive equilibrium. The condition rules out for example the case that for a trade the seller would never sell under any price and the buyer would buy under any price. Fleiner et al. (2018) also introduce a stronger regularity condition such that under this condition and (weak) full substitutability, competitive equilibrium outcomes are equivalent to trail stable outcomes which is a cooperative solution concepts that generalizes pairwise stability from matching markets. Their condition is the following:

Bounded willingness to pay (BWP): The utility function u^f satisfies bounded willingness to pay if there exists a $K > 0$ such that for all $p \in \mathbb{R}^\Omega$ and $\Psi \in D^f(p)$ if $\omega \in \Psi_{\to f}$ then $p_\omega < K$ and if $\omega \in \Psi_{f\to}$ then $p_\omega > -K$.

Next we show that under the assumption of BWP and an additional assumption that we call "no undesired trades", the set of equilibrium price vectors is compact. "No undesired trades" requires that for each trade there is a (high enough) price such that the seller would like to execute the trade and a (low enough) price such that the buyer would like to execute the trade.[7]

No undesired trades (NUT): The utility function u^f satisfies no undesired trades if for each $\omega \in \Omega_f$ and $p_{-\omega} \in \mathbb{R}^{\Omega \setminus \{\omega\}}$ there exists a $p_\omega \in \mathbb{R}$ and $\Psi \in D^f(p)$ with $\omega \in \Psi$.

Next we show that under the assumption of BWP and NUT, the set of equilibrium price vectors is compact. As a corollary of this result and the lattice result of the previous section, we obtain the result that there exist extremal points in the set of equilibrium vectors, provided that BWP, NUT, FS and LAD/LAS are satisfied.

Proposition 1. *Under the assumption of BWP and NUT the set of competitive equilibrium vectors is a compact set.*

Proof. Following a proof idea originally introduced by Kelso and Crawford (1982), we can characterize competitive equilibria by a zero-surplus condition: Define a surplus function $\mathbb{R}^\Omega \to \mathbb{R}$ by

$$Z(p) := \min_{\Psi \subseteq \Omega} \max_{f \in F} \max_{\Psi' \subseteq \Omega_f} u^f(\Psi', p) - u^f(\Psi, p).$$

By definition, for each $f \in F$, we have $D^f(p) = \text{argmax}_{\Psi' \subseteq \Omega_f} u^f(\Psi', p)$. Thus for each arrangement $[\Psi, p]$, we have $\max_{f \in F} \max_{\Psi' \subseteq \Omega_f} u^f(\Psi', p) - u^f(\Psi, p) \geq 0$ with equality if and only if $\Psi \in \mathcal{E}(u, p)$. Thus $\mathcal{E}(u, p) \neq \emptyset$ if and only if $Z(p) = 0$.

Now note that the surplus function is continuous, as $u^f(\Psi', p) - u^f(\Psi, p)$ is continuous in p and the maximum resp. minimum of finitely many continuous functions is continuous. Thus the set of competitive equilibrium vectors is closed as it is the pre-image of the closed set $\{0\}$ under the continuous map Z. Moreover, by BWP and NUT, $\mathcal{E}(u)$ is bounded. Hence $\mathcal{E}(u)$ is compact. $\square$

Corollary 1. *Under the assumption of BWP, NUT, FS, LAD and LAS, the set of competitive equilibrium vectors is a bounded lattice where the maximal element is given by*

$$\bar{p} = (\sup\{p_\omega : p \in \mathcal{E}(u)\})_{\omega \in \Omega},$$

and the minimal element is given by

$$\underline{p} = (\inf\{p_\omega : p \in \mathcal{E}(u)\})_{\omega \in \Omega}.$$

[7] The assumption is for example satisfied if a utility function satisfies a full range assumption, i.e. if for each $\Psi \subseteq \Omega$ the function $u^f(\Psi, \cdot)$ is a surjective function from $\mathbb{R}^\Psi$ to $\mathbb{R}$. The full range assumption is made by Demange and Gale (1985) to obtain similar results than ours for one-to-one matching markets with transfers.

Proof. By Theorem 1, $\mathcal{E}(u)$ is compact and by Corollary 2 in Fleiner et al. (2018), $\mathcal{E}(u) \neq \emptyset$. Thus for each $\omega \in \Omega$, the set $\{p_\omega \in \mathbb{R} : p \in \mathcal{E}(u)\}$ is compact and non-empty as well. Thus for each $\omega \in \Omega$ there exists a $p^\omega \in \mathcal{E}(u)$ with $p^\omega_\omega = \max\{p_\omega : p \in \mathcal{E}(u)\}$. Taking the join of the finite set of vectors $(p^\omega)_{\omega\in\Omega}$ yields an equilibrium price vector, which by definition is $\bar{p}$. A similar argument establishes that $\underline{p} \in \mathcal{E}(u)$. □

If we drop NUT, the set of equilibrium vectors can fail to be bounded for trades that are never realized under any prices. However, equilibrium prices are still bounded for all other trades. In particular, we can obtain the result that there exist an equilibrium that is a most preferred equilibrium for all terminal buyers and an equilibrium that is a most preferred equilibrium allocations for all terminal sellers.

Theorem 2 (Existence of Extremal Equilibria). *Under the assumption of BW, FS, LAD and LAS, there exists a seller optimal equilibrium, i.e. a* $[\bar{\Psi}, \bar{p}]$ *with* $\bar{\Psi} \in \mathcal{E}(u, p)$ *such that for each terminal seller* $f \in F$*:*

$$u^f(\bar{\Psi}, \bar{p}) \geq u^f(\Psi, p) \text{ for each } [\Psi, p] \text{ with } \Psi \in \mathcal{E}(u, p),$$

and a buyer optimal equilibrium, i.e. a $[\underline{\Psi}, \underline{p}]$ *with* $\underline{\Psi} \in \mathcal{E}(u, \underline{p})$ *such that for each terminal buyer* $f \in F$*:*

$$u^f(\underline{\Psi}, \underline{p}) \geq u^f(\Psi, p) \text{ for each } [\Psi, p] \text{ with } \Psi \in \mathcal{E}(u, p).$$

Proof. By BWP there is a $K > 0$ such that for each $f \in F$ and each $p \in \mathbb{R}^\Omega$, if $\Psi \in D^f(p)$ then $p_\omega < K$ for $\omega \in \Omega_{\to f}$ and $p_\omega > -K$ for $\omega \in \Omega_{f\to}$. Thus for each equilibrium $[\Psi, p]$ there is a $p' \in [-K, K]^\Omega$ with $\Psi \in \mathcal{E}(u, p')$ and $p'_\omega = p_\omega$ for each $\omega \in \Psi$. Observe that the argument for establishing that $\mathcal{E}(u)$ is closed in the proof of Proposition 1 only depended on BWP. Thus $\mathcal{E}(u)$ is closed and $\mathcal{E}'(u) := \mathcal{E}(u) \cap [-K, K]^\Omega$ is compact. Now using the same argument as in the proof of Corollary 1, we can show that

$$\bar{p} = (\sup\{p_w : p \in \mathcal{E}'(u)\})_{\omega\in\Omega} \in \mathcal{E}'(u),$$

and

$$\underline{p} = (\inf\{p_w : p \in \mathcal{E}'(u)\})_{\omega\in\Omega} \in \mathcal{E}'(u).$$

Since for each $p \in \mathcal{E}(u)$ and $\Psi \in \mathcal{E}(u, p)$ there is a $p' \in \mathcal{E}'(u)$ with $p'_\omega = p_\omega$ for $\omega \in \Psi$ and $\Psi \in \mathcal{E}(u, p')$, this concludes the proof. □

Remark 2. Fleiner et al. (2018) establish that under the assumption of BWP and (weak) FS, equilibrium allocations are equivalent to trail-stable allocations. Thus, alternatively the proposition could be stated in the form that under BWP, FS, LAD, LAS there is a seller optimal trail stable allocation and a buyer optimal trail stable allocation.

3.3 Strategic Considerations

The existence of buyer (seller) optimal equilibria established in Proposition 2, allows to obtain a group-incentive compatibility result. In the following, a domain of utility functions is a set $\mathcal{U} = \times_{f \in F} \mathcal{U}_f$ where $\mathcal{U}_f$ is a set of (continuous and monotonic) utility functions for firm f. A **mechanism** is a function $\mathcal{M} : \mathcal{U} \to \mathcal{A}$. A mechanism is **(weakly) group-strategy-proof** for a set of workers $F' \subseteq F$ on the domain $\mathcal{U}' \subseteq \mathcal{U}$ if for each $u, \tilde{u} \in \mathcal{U}'$ with $\tilde{u}^{-F'} = u^{-F'}$, there exist a $f \in F'$ with

$$u^f(\mathcal{M}(u)) \geq u^f(\mathcal{M}(\tilde{u})).$$

Theorem 2 allows to define a class of focal mechanisms on the domain of utility profiles satisfying BWP, FS, LAD and LAS: a **seller-optimal mechanism** maps to each utility profile a seller optimal equilibrium allocation and a **buyer-optimal mechanism** maps to each utility profile a seller optimal equilibrium allocation.

To obtain a group-strategy-proofness results for terminal buyers (sellers) for the buyer (seller)-optimal mechanism, we have to restrict the domain. In the following a **unit demand** utility function is a u^f such that for the induced demand D^f at each $p \in \mathbb{R}^\Omega$ and $\Psi \in D^f(p)$ we have $|\Psi_{\to f}| \leq 1$ and a **unit supply** utility function is a u^f such that for the induced demand D^f at each $p \in \mathbb{R}^\Omega$ and $\Psi \in D^f(p)$ we have $|\Psi_{f \to}| \leq 1$

To establish the group-strategy-proofness for terminal buyers (sellers), we simplify and adapt to our context an argument introduced by Hatfield and Kojima (2009) for matching with contracts.[8] As observed by Jagadeesan et al. (2018), the argument of Hatfield and Kojima (2009) relies crucially on the possibility of reporting preferences with income effects. Hence working with the larger domain of continuous and monotonic utility functions instead of quasi-linear utility functions is crucial for the argument. However, our result implies (a forteriori) that a buyer (seller) optimal mechanism is also group-strategyproof on the domain of quasi-linear utility functions such that terminal buyers (sellers) have unit demand and all other firms have FS demand.

Theorem 3. *Each buyer-optimal (seller-optimal) mechanism is group-strategy-proof for terminal buyers (sellers) on the domain of utility profiles such that terminal buyers' (sellers') utility functions satisfy Unit Demand (Supply) and BWP and all other firms' utility functions satisfy BWP, FS, LAD and LAS.*

[8] The mechanism falls outside of the domain defined by Barberà et al. (2016) on which strategy-proofness is equivalent to (weak)-group-strategy-proofness. Thus, it is not sufficient to only show strategy-proofness and invoke the result of Barberà et al. (2016).

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A Proof of Lemma 1

Proof. Let $p \in \mathbb{R}^\Omega$. Let $\epsilon > 0$ such that

$$\epsilon < \min_{\Psi \in D^f(p), \Psi' \notin D^f(p)} u^f(\Psi, p) - u^f(\Psi', p).$$

By continuity of u^f in prices, for each $\Psi, \Psi' \subseteq \Omega_f$ the function

$$G_{\Psi,\Psi'}(p) := u^f(\Psi, p) - u^f(\Psi', p)$$

is continuous and there exists a $\delta_{\Psi,\Psi'} > 0$ such that for $p \in \mathbb{R}^{\Omega_f}$ with $\|p - p'\| < \delta_{\Psi,\Psi'}$ we have $|G_{\Psi,\Psi'}(p) - G_{\Psi,\Psi'}(p')| < \epsilon$. Define $\delta > 0$ by

$$\delta := \min_{\Psi \in D^f(p), \Psi' \notin D^f(p)} \delta_{\Psi,\Psi'}.$$

Let $\Psi \in D^f(p)$ and $\Psi' \notin D^f(p)$. Let $p' \in \mathbb{R}^{\Omega_f}$ with $\|p - p'\| < \delta$. By construction $|G_{\Psi,\Psi'}(p) - G_{\Psi,\Psi'}(p')| < \epsilon$, and $G_{\Psi,\Psi'}(p) > \epsilon$. Therefore $G_{\Psi,\Psi'}(p') > 0$ and $u^f(\Psi, p') > u^f(\Psi', p')$. Thus, each bundle of workers that is not utility maximizing under p is also not utility maximizing under p'. We have $D_f(p') \subseteq D_f(p)$. □

We repeatedly make use of perturbation arguments, where in the case of multi-valued demand we slightly perturb prices to obtain a price vector where the demand is single-valued and selects from the demand at the unperturbed price vector. The following lemma allows us to use this argument and follows from Lemma 1.

Lemma 3. *Let $p \in \mathbb{R}^{\Omega_f}$. For every $\epsilon_0 > 0$ there is an $\epsilon \in \mathbb{R}^{\Omega_f}$ with $0 < \epsilon_\omega < \epsilon_0$ for $\omega \in \Omega_{\to f}$ and $-\epsilon_0 < \epsilon_\omega < 0$ for $\omega \in \Omega_{f\to}$ such that $|D^f(p + \epsilon)| = 1$ and $\Psi \in D^f(p)$ for the unique $\Psi \in D^f(p + \epsilon)$.*

Proof. Let

$$\Phi(p) := \{\omega \in \Omega : \exists \Psi \in D^f(p) \text{ and } \Psi' \in D^f(p) \text{ with } \omega \in \Psi, \omega \notin \Psi'\}.$$

We prove the lemma by induction on $|\Phi(p)|$. For each $p \in \mathbb{R}^\Omega$ with $|\Phi(p)| = 0$ the demand is single-valued and by Lemma 1 we can select a $\epsilon \in \mathbb{R}^{\Omega_f}$ as desired.

Now let $k > 0$ and suppose for each $p \in \mathbb{R}^{\Omega_f}$ with $|\Phi(p)| \leq k$ there is for each $\epsilon_0 > 0$ an $\epsilon \in \mathbb{R}^{\Omega_f}$ with $0 < \epsilon_\omega < \epsilon_0$ for $\omega \in \Omega_{\to f}$ and $-\epsilon_0 < \epsilon_\omega < 0$ for $\omega \in \Omega_{f\to}$ such that $|D^f(p + \epsilon)| = 1$ and $\Psi \in D^f(p)$ for the unique $\Psi \in D^f(p + \epsilon)$. Now let $p' \in \mathbb{R}^{\Omega_f}$ with $|\Phi(p')| = k + 1$. Let $\epsilon'_0 > 0$. Choose an arbitrary $\tilde{\omega} \in \Phi(p')$. By Lemma 1, there exists a $\epsilon_1 > 0$ such that for $q \in \mathbb{R}^{\Omega_f}$ with $\|q - p\| < \epsilon_1$ we have $D^f(q) \subseteq D^f(p)$. We may choose $\epsilon_1 < \epsilon'_0/2$. Let $\tilde{\epsilon} \in \mathbb{R}^{\Omega_f}$, be defined by $\tilde{\epsilon}_{\tilde{\omega}} = \epsilon_1$ if $\tilde{\omega} \in \Omega_{\to f}$, resp. $\tilde{\epsilon}_{\tilde{\omega}} = -\epsilon_1$ if $\tilde{\omega} \in \Omega_{f\to}$, and $\tilde{\epsilon}_\omega = 0$ for $\omega \neq \tilde{\omega}$. As u^f is monotonic, we have $|D^f(p' + \tilde{\epsilon})| \leq k$. Thus by the induction assumption with $\epsilon_0 = \epsilon'_0/2$ and $p = p' + \tilde{\epsilon}$ there is a $\epsilon \in \mathbb{R}^{\Omega_f}$ with $0 < \epsilon_\omega < \epsilon_0$ for $\omega \in \Omega_{\to f}$ and $-\epsilon_0 < \epsilon_\omega < 0$ for $\omega \in \Omega_{f\to}$ such that $|D^f(p' + \tilde{\epsilon} + \epsilon)| = 1$ and $\Psi \in D^f(p' + \tilde{\epsilon}) \subseteq D^f(p')$ for the unique $\Psi \in D^f(p' + \epsilon + \tilde{\epsilon})$. Thus we can choose $\epsilon' = \epsilon + \tilde{\epsilon}$. □

B Different Versions of Full Substitutability

In Appendix A of Hatfield et al. (2018b), the authors introduce the contraction and expansion version of full substitutability that differ in regard to how they are defined at price vectors where the demand is multi-valued. Note that full substitutability relates the demand at two price vectors p and p'. If $p_\omega \leq p'_\omega$ for $\omega \in \Omega_{f\to}$ and $p'_\omega = p_\omega$ for $\omega \in \Omega_{\to f}$ respectively if $p_\omega \geq p'_\omega$ for $\omega \in \Omega_{\to f}$ and $p'_\omega = p_\omega$ and $\omega \in \Omega_{f\to}$, then the choice set expands from p' to p, and contracts from p to p'. Now the contraction version of full substitutability requires that "for all $\Psi \in D^f(p)$ there is a $\Psi' \in D^f(p')$ such that..." whereas the expansion version inverts the order of quantification and requires that "for all $\Psi' \in D^f(p')$ there is a $\Psi \in D^f(p)$ such that..." We further split full substituability into same-side substitutability (SSS) and cross-side complementarity (CSC) and the laws of aggregate demand and supply, all of which are implied by all versions of full substitutability in the quasi-linear case. We use the expansion version of SSS as our main definition. Alternatively, we can consider the contraction version.

Contraction Same-Side Substitutability: For $p, p' \in \mathbb{R}^\Omega$ and each $\Psi \in D^f(p)$ there exists a $\Psi' \in D^f(p')$ such that if $p_\omega = p'_\omega$ for $\omega \in \Omega_{f\to}$ and $p_\omega \leq p'_\omega$ for $\omega \in \Omega_{\to f}$, then

$$\{\omega \in \Psi_{\to f} : p_\omega = p'_\omega\} \subseteq \Psi'_{\to f},$$

and if $p_\omega = p'_\omega$ for $\omega \in \Omega_{\to f}$ and $p_\omega \geq p'_\omega$ for $\omega \in \Omega_{f\to}$, then

$$\{\omega \in \Psi_{f\to} : p_\omega = p'_\omega\} \subseteq \Psi'_{f\to}.$$

We have previously introduced weak full substitutability that can be further decomposed into weak SSS and weak CSC.

Weak Same-Side Substitutability: For $p, p' \in \mathbb{R}^\Omega$ such that $D^f(p) = \{\Psi\}$ and $D^f(p') = \{\Psi'\}$ if $p_\omega = p'_\omega$ for $\omega \in \Omega_{f\to}$ and $p_\omega \leq p'_\omega$ for $\omega \in \Omega_{\to f}$, then

$$\{\omega \in \Psi_{\to f} : p_\omega = p'_\omega\} \subseteq \Psi'_{\to f},$$

and if $p_\omega = p'_\omega$ for $\omega \in \Omega_{\to f}$ and $p_\omega \geq p'_\omega$ for $\omega \in \Omega_{f\to}$, then

$$\{\omega \in \Psi_{f\to} : p_\omega = p'_\omega\} \subseteq \Psi'_{f\to}.$$

All three notions of SSS, the weak, the contraction version, as well as the expansion version that we use as our main definition are equivalent under quasi-linear utility, as shown by Hatfield et al. (2018b). Fleiner et al. (2018) show that the weak and the expansion versions are equivalent for general utility functions (they establish an equivalence for the notions of "decreasing-price full substitutability for sales" and "increasing-price full substitutability for purchases" whose combinations is equivalent to expansion SSS).

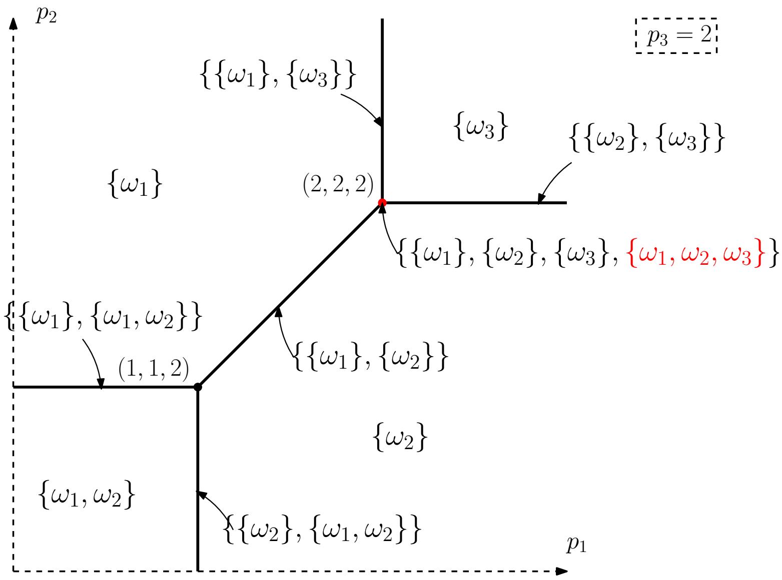

Figure 2: The demand in price space for $p_3 = 2$.

Proposition 2 (Hatfield et al., 2018b; Fleiner et al., 2018)**.** *Let u^f be a monotonic and continuous utility function with induced demand D^f.*

1. *D^f satisfies weak SSS if and only if it satisfies (Expansion) SSS.*
2. *If D^f satisfies Contraction SSS then it satisfies weak SSS.*
3. *IF u^f is quasi-linear and D^f satisfies weak SSS, then D^f satisfies Contraction SSS.*

In general weak SSS does not imply Contraction SSS as the following example shows. See Figure 2 for a geometric representation of the demand in the example.

Example 2. Consider three trades $\Omega = \{\omega_1, \omega_2, \omega_3\}$ with $f = b(\omega_1) = b(\omega_2) = b(\omega_3)$. We let $u^f(\emptyset) = 0$, $u^f(\{\omega_i\}, p_i) = 3 - p_i$ for $i = 1, 2, 3$, $u^f(\{\omega_i, \omega_j\}, p_i, p_j) = 4 - p_i - p_j$ for $i \neq j$ and

$$u^f(\{\omega_1,\omega_2,\omega_3\},p) = \begin{cases} 4 - p_1 - p_2 - p_3 & \text{if } p_1 + p_2 + p_3 \leq 0 \\ 4 - 3\sqrt{\frac{p_1+p_2+p_3}{6}} & \text{if } 6 \geq p_1 + p_2 + p_3 > 0, \\ 7 - p_1 - p_2 - p_3 & \text{else} \end{cases}$$

Observe that

$$D^f(2,2,2) = \{\{\omega_1,\omega_2,\omega_3\},\{\omega_1\},\{\omega_2\},\{\omega_3\}\}$$

but

$$D^f(3,2,2) = \{\{\omega_2\},\{\omega_3\}\}.$$

As $\{\omega_1, \omega_2, \omega_3\} \in D^f(2,2,2)$, Contraction SSS would require that there is a $\Psi \in D^f(3,2,2)$ with $\{\omega_2, \omega_3\} \subseteq \Psi$. Hence Contraction SSS is not satisfied. As the demand at $(2,2,2)$ and $(3,2,2)$ is multi-valued, Weak SSS does not impose any structure here. More generally, note that if we replace u^f by the quasi-linear utility functions $\tilde{u}^f$ such that $\tilde{u}^f(\{\omega_1, \omega_2, \omega_3\}, p) = 4 - p_1 - p_2 - p_3$ for all $p \in \mathbb{R}^\Omega$ and u^f remains otherwise unchanged, only the demand at prices $(2,2,2)$ changes. One readily checks that $\tilde{u}^f$ satisfies (Weak) SSS. Hence u^f satisfies Weak SSS. □

Similarly as for Same-Side Substitutabilty, we can define an alternative version of Cross-Side Complementarity.

Contraction Cross-Side Complementarity: For each $\Psi \in D^f(p)$ there exists a $\Psi' \in D^f(p')$ such if $p_\omega = p'_\omega$ for $\omega \in \Omega_{f\to}$ and $p_\omega \leq p'_\omega$ for $\omega \in \Omega_{\to f}$, then

$$\Psi'_{f\to} \subseteq \Psi_{f\to},$$

and if $p_\omega = p'_\omega$ for $\omega \in \Omega_{\to f}$ and $p_\omega \geq p'_\omega$ for $\omega \in \Omega_{f\to}$, then

$$\Psi'_{\to f} \subseteq \Psi_{\to f}.$$

As before, we also define a weak version of CSC that together with Weak SSS defines Weak Full Substitutability.

Weak Cross-Side Complementarity: For each $p, p' \in \mathbb{R}^\Omega$ with $D^f(p) = \{\Psi\}$ and $D^f(p') = \{\Psi'\}$ such if $p_\omega = p'_\omega$ for $\omega \in \Omega_{f\to}$ and $p_\omega \leq p'_\omega$ for $\omega \in \Omega_{\to f}$, then

$$\Psi'_{f\to} \subseteq \Psi_{f\to},$$

and if $p_\omega = p'_\omega$ for $\omega \in \Omega_{\to f}$ and $p_\omega \geq p'_\omega$ for $\omega \in \Omega_{f\to}$, then

$$\Psi'_{\to f} \subseteq \Psi_{\to f}.$$

Similarly as for SSS, the weak, the contraction version as well as the expansion version of CSC that we use as our main definition are equivalent under quasi-linear utility, as shown by Hatfield et al. (2018b). Fleiner et al. (2018) show that the weak and the contraction versions are equivalent for general preferences (they establish an equivalence for the notions of "increasing-price full substitutability for sales" and "decreasing-price full substitutability for purchases" whose combinations is equivalent to contraction CSC).

Proposition 3 (Hatfield et al., 2018b; Fleiner et al., 2018)**.** *Let u^f be a monotonic and continuous utility function with induced demand D^f.*

1. *D^f satisfies weak CSC if and only if it satisfies Contraction CSC.*

2. *If D^f satisfies (Expansion) CSC then it satisfies Weak CSC.*

3. *IF u^f is quasi-linear and D^f satisfies Weak CSC, then D^f satisfies (Expansion) CSC.*

In general weak CSC does not imply (expansion) CSC as our Example 1 shows.

C Proof of Lemma 2

Proof. By monotonicity of u^f and upper hemi-continuity of D^f it suffices to consider the case that $p'_\omega = p_\omega$ for $\omega \in \Omega_f \setminus \Psi'$ and $p'_\omega \neq p_\omega$ for $\omega \in \Psi'$.

Let $0 < \epsilon_0 < \min_{\omega\in\Psi'} |p'_\omega - p_\omega|$. By Lemma 3, there exists an an $\epsilon \in \mathbb{R}^\Omega$ with $0 < \epsilon_\omega < \epsilon_0$ for $\omega \in \Omega_{\to f}$ and $-\epsilon_0 < \epsilon_\omega < 0$ for $\omega \in \Omega_{f\to}$ such that $|D^f(p+\epsilon)| = 1$ and $\Psi \in D^f(p)$ for the unique $\Psi \in D^f(p+\epsilon)$. Define

$$\epsilon'_\omega := \begin{cases} \epsilon_\omega, & \text{if} \quad \omega \in \Omega_f \setminus \Psi', \\ 0, & \text{else.} \end{cases}$$

Observe that by monotonicity of u^f for each $\Xi \subseteq \Omega_f$ we have $u^f(\Xi, p'+\epsilon') \leq u^f(\Xi, p')$. Moreover, as $\epsilon'_\omega = 0$ for $\omega \in \Psi'$, we have $u^f(\Psi', p'+\epsilon') = u^f(\Psi', p')$. Thus $\Psi' \in D^f(p'+\epsilon')$. Moreover, by monotonicity of u^f for each $\Xi \subseteq \Omega_f$ with $\Xi \setminus \Psi' \neq \emptyset$, we have $u^f(\Xi, p'+\epsilon') < u^f(\Xi, p') \leq u^f(\Psi', p') = u^f(\Psi', p'+\epsilon')$ and therefore $\Xi \notin D^f(p'+\epsilon')$. Therefore for each $\Xi \in D^f(p'+\epsilon')$, we have $\Xi \subseteq \Psi'$.

Now consider the unique $\Psi \in D^f(p+\epsilon)$. We will show that $\Psi' = \Psi$. As $\Psi \in D^f(p)$ this will prove the first part of the lemma. Let $\tilde{p} \in \mathbb{R}^\Omega$ be defined by

$$\tilde{p}_\omega := \begin{cases} p'_\omega + \epsilon'_\omega \text{ for } \omega \in \Omega_{f\to}, \\ p_\omega + \epsilon_\omega \text{ else.} \end{cases}$$

First we show the following:

Claim 1. *There exists a $\tilde{\Psi} \in D^f(\tilde{p})$ with $\tilde{\Psi}_{\to f} \subseteq \Psi'_{\to f}$ and $\Psi'_{f\to} \subseteq \tilde{\Psi}_{f\to}$.*

Proof. As $\tilde{p}_\omega = p_\omega + \epsilon_\omega = p'_\omega + \epsilon'_\omega$ for $\omega \in \Omega_{\to f} \setminus \Psi'$, SSS implies that there is a $\bar{\Psi} \in D^f(\tilde{p})$ with $\bar{\Psi}_{\to f} \subseteq \Psi'_{\to f}$. Let $\tilde{\epsilon} \in \mathbb{R}^\Omega$ with $\tilde{\epsilon}_\omega > 0$ for $\omega \in \Omega_{\to f} \setminus \Psi'_{\to f}$ and $\epsilon_\omega = 0$ otherwise. By monotonicity of u^f, for each $\tilde{\Psi} \subseteq \Omega_f$ with $\tilde{\Psi}_{\to f} \not\subseteq \Psi'_{\to f}$ we have $u^f(\tilde{\Psi}, \tilde{p}+\tilde{\epsilon}) < u^f(\tilde{\Psi}, \tilde{p})$ and for each $\tilde{\Psi} \subseteq \Omega_f$ with $\tilde{\Psi}_{\to f} \subseteq \Psi'_{\to f}$ we have $u^f(\tilde{\Psi}, \tilde{p}+\tilde{\epsilon}) = u^f(\tilde{\Psi}, \tilde{p})$. Thus, as $\bar{\Psi} \in D^f(\tilde{p})$ and $\bar{\Psi}_{\to f} \subseteq \Psi'_{\to f}$, for each $\tilde{\Psi} \in D^f(\tilde{p}+\tilde{\epsilon})$ we have $\tilde{\Psi}_{\to f} \subseteq \Psi'_{\to f}$ and $\tilde{\Psi} \in D^f(\tilde{p})$.

By monotonicity of u^f, for each $\tilde{\Psi} \subseteq \Omega_f$, we have $u^f(\tilde{\Psi}, p'+\epsilon'+\tilde{\epsilon}) \leq u^f(\tilde{\Psi}, p'+\epsilon')$ and we have $u^f(\Psi', p'+\epsilon'+\tilde{\epsilon}) = u^f(\Psi', p'+\epsilon')$. Thus $\Psi' \in D^f(p'+\epsilon'+\tilde{\epsilon})$. As $\Psi' \in D^f(p'+\epsilon'+\tilde{\epsilon})$, CSC implies that there is a $\tilde{\Psi} \in D^f(\tilde{p}+\tilde{\epsilon})$ with $\Psi'_{f\to} \subseteq \tilde{\Psi}_{f\to}$. As previously observed, we have $\tilde{\Psi}_{\to f} \subseteq \Psi'_{\to f}$ and $\tilde{\Psi} \in D^f(\tilde{p})$. □

With the claim we can prove the first part of the lemma. Let $\tilde{\Psi} \in D^f(\tilde{p})$ as in the claim. As $\Psi'_{f\to} \subseteq \tilde{\Psi}_{f\to}$ and $\tilde{p}_\omega = p'_\omega + \epsilon'_\omega = p_\omega + \epsilon_\omega$ for $\omega \in \Omega_{f\to} \setminus \Psi'$, SSS implies $\Psi_{f\to} \subseteq \tilde{\Psi}_{f\to}$. By CSC, we have $\tilde{\Psi}_{\to f} \subseteq \Psi_{\to f}$. By LAS we have

$$|\Psi_{f\to}| - |\Psi_{\to f}| \geq |\tilde{\Psi}_{f\to}| - |\tilde{\Psi}_{\to f}|.$$

Thus $\tilde{\Psi} = \Psi$. By LAD and LAS we have

$$|\Psi_{\to f}| - |\Psi_{f\to}| \geq |\Psi'_{\to f}| - |\Psi'_{f\to}|.$$

Combining this with the observation that $\Psi_{\to f} = \tilde{\Psi}_{\to f} \subseteq \Psi'_{\to f}$ and $\Psi'_{f\to} \subseteq \tilde{\Psi}_{f\to} = \Psi_{f\to}$, we have $\Psi' = \Psi$.

Now we show that $D^f(p) = \{\Psi'\}$ if all of the inequalities are strict. Suppose there is a $\Xi \neq \Psi'$ with $\Xi \in D^f(p)$. Then there is a $\tilde{\omega} \in \Omega_f$ with $\tilde{\omega} \in \Xi \setminus \Psi$ or $\tilde{\omega} \in \Psi \setminus \Xi$. In the first case, let $\tilde{p} \in \mathbb{R}^{\Omega_f}$ with $\tilde{p}_{\tilde{\omega}} = p'_{\tilde{\omega}}$ and $\tilde{p}_\omega = p_\omega$ for $\omega \neq \tilde{\omega}$. Note that $\Psi' \in D^f(\tilde{p})$. Thus, by monotonicity, we have $u^f(\Xi, p) < u^f(\Xi, \tilde{p}) \leq u^f(\Psi', \tilde{p}) = u^f(\Psi', p)$ contradicting the assumption that $\Xi \in D^f(p)$. In the second case, let $\tilde{p} \in \mathbb{R}^{\Omega_f}$ with $\tilde{p}_{\tilde{\omega}} = p'_{\tilde{\omega}}$ and $\tilde{p}_\omega = p_\omega$ for $\omega \neq \tilde{\omega}$. Note that $\Psi' \in D^f(\tilde{p})$. Thus, by monotonicity, we have $u^f(\Xi, p) = u^f(\Xi, \tilde{p}) \leq u^f(\Psi', \tilde{p}) < u^f(\Psi', p)$ contradicting the assumption that $\Xi \in D^f(p)$. □

D Proof of Theorems 1

The proof relies on the following two lemmata.

Lemma 4. *Let u^f be a utility function inducing a demand correspondence D^f satisfying full substitutability and the laws of aggregate demand and supply. Let $P \subseteq \mathbb{R}^{\Omega_f}$ be finite. Then there is a (single-valued) demand function $\tilde{D}^f : P \to 2^{\Omega_f}$ that selects from D^f, i.e. $\tilde{D}^f(p) \in D^f(p)$ for $p \in P$ and satisfies full substitutability and the laws of aggregate demand and supply.*

Proof. By Lemma 1, there exists an $\epsilon_0 > 0$ such that for each $p \in P$ and every q with $\|q - p\| < \epsilon_0$ we have $D^f(q) \subseteq D^f(p)$. Let $P = \{p^1, \ldots, p^n\}$. By Lemma 3, there is a $\epsilon^1 \in \mathbb{R}^{\Omega_f}$ with $\|\epsilon^1\| < \epsilon_0$ such that $|D^f(p^1 + \epsilon^1)| = 1$ and $\Psi \in D^f(p^1)$ for the unique $\Psi \in D^f(p^1 + \epsilon^1)$. Consider $P^1 := \{p^1 + \epsilon^1, \ldots, p^n + \epsilon^1\}$. For each $i = 1, \ldots, n$ we have $D^f(p^i + \epsilon^1) \subseteq D^f(p^i)$. By Lemma 1, there exists an $\epsilon_1 > 0$ such that for each $p \in P^1$ and every q with $\|q - p\| < \epsilon_1$ we have $D^f(q) \subseteq D^f(p)$. By Lemma 3, there is a $\epsilon^2 \in \mathbb{R}^{\Omega_f}$ with $\|\epsilon^2\| < \epsilon_1$ such that $|D^f(p^2 + \epsilon^1 + \epsilon^2)| = 1$ and $\Psi \in D^f(p^2 + \epsilon^2)$ for the unique $\Psi \in D^f(p^2 + \epsilon^1)$. Next consider $P^2 := \{p^1 + \epsilon^1 + \epsilon^2, \ldots, p^n + \epsilon^1 + \epsilon^2\}$. For each $i = 1, \ldots, n$ we have $D^f(p^i + \epsilon^1 + \epsilon^2) \subseteq D^f(p^i + \epsilon^1) \subseteq D^f(p^i)$ and so on. Iterating in this way, we obtain $\epsilon^1, \ldots, \epsilon^n$ such that for each $i = 1, \ldots, n$, we have $|D^f(p^i + \sum_{j=1}^n \epsilon^j)| = 1$

and $\Psi^i \in D^f(p^i)$ for the unique $\Psi^i \in D^f(p^i + \sum_{j=1}^n \epsilon^j) \subseteq D^f(p^i)$. We define $\tilde{D}^f(p^i) = \Psi^i$. By construction $\tilde{D}^f(p^i) \in D^f(p^i)$. Moreover, as all price vectors are translated by the same vector $\sum_{j=1}^n \epsilon^j$, full substitutability and the laws of aggregate demand and supply are inherited from D^f. □

Lemma 5. *Let $p, p' \in \mathbb{R}^{\Omega_f}$ and define $\bar{p}, \underline{p} \in \mathbb{R}^{\Omega_f}$ by*

$$\bar{p}_\omega := \max\{p_\omega, p'_\omega\}, \quad \underline{p}_\omega := \min\{p_\omega, p'_\omega\}.$$

Let $\Psi \in D^f(p)$ and $\Psi' \in D^f(p')$.

1. *There is a $\bar{\Psi} \in D^f(\bar{p})$ with*

$$\{\omega \in \Psi_{\to f} : p_\omega \geq p'_\omega\} \cup \{\omega \in \Psi'_{\to f} : p'_\omega > p_\omega\} \subseteq \bar{\Psi}_{\to f},$$
$$\bar{\Psi}_{f\to} \subseteq \{\omega \in \Psi_{f\to} : p_\omega \geq p'_\omega\} \cup \{\omega \in \Psi'_{f\to} : p'_\omega > p_\omega\}.$$

2. *There is a $\underline{\Psi} \in D^f(\underline{p})$ with*

$$\underline{\Psi}_{\to f} \subseteq \{\omega \in \Psi_{\to f} : p'_\omega \geq p_\omega\} \cup \{\omega \in \Psi'_{\to f} : p_\omega > p'_\omega\},$$
$$\{\omega \in \Psi_{f\to} : p'_\omega \geq p_\omega\} \cup \{\omega \in \Psi'_{f\to} : p_\omega > p'_\omega\} \subseteq \underline{\Psi}_{f\to}.$$

3. *$\bar{\Psi}$ and $\underline{\Psi}$ can be chosen such that*

$$|\underline{\Psi}_{\to f}| - |\underline{\Psi}_{f\to}| \geq |\Psi_{\to f}| - |\Psi_{f\to}| \geq |\bar{\Psi}_{\to f}| - |\bar{\Psi}_{f\to}|.$$

Proof. By Lemma 1, there exists an $\epsilon_0 > 0$ such that for each $q \in \{p, p', \bar{p}, \underline{p}\}$ and every $\tilde{q}$ with $\|\tilde{q} - q\| < \sqrt{|\Omega_f|} \cdot \epsilon_0$ we have $D^f(\tilde{q}) \subseteq D^f(q)$. We may choose $\epsilon_0 < \min_{\omega \in \Omega : p'_\omega \neq p_\omega} |p'_\omega - p_\omega|$.

Define $\epsilon' \in \mathbb{R}^{\Omega_f}$ by

$$\epsilon'_\omega = \begin{cases} \epsilon_0, & \text{if } \omega \in \Psi'_{f\to} \text{ and } p'_\omega \neq p_\omega, \\ -\epsilon_0, & \text{if } \omega \in \Omega_{f\to} \setminus \Psi' \text{ and } p'_\omega \neq p_\omega, \\ -\epsilon_0, & \text{if } \omega \in \Psi'_{\to f} \text{ and } p'_\omega \neq p_\omega, \\ \epsilon_0, & \text{if } \omega \in \Omega_{\to f} \setminus \Psi' \text{ and } p'_\omega \neq p_\omega, \\ 0, & \text{if } p'_\omega = p_\omega. \end{cases}$$

First we prove the following claim.

Claim 2. *For each $\Xi \in D^f(p' + \epsilon')$ we have $\{\omega \in \Psi' : p'_\omega \neq p_\omega\} \subseteq \Xi$ and $\{\omega \notin \Psi' : p'_\omega \neq p_\omega\} \cap \Xi = \emptyset$.*

Proof. First we show that for each $\Xi \in D^f(p'+\epsilon')$ we have $\{\omega \in \Psi' : p'_\omega \neq p_\omega\} \subseteq \Xi$. Suppose not, and there is a $\Xi \in D^f(p'+\epsilon')$ and a $\tilde{\omega} \in \{\omega \in \Psi' : p'_\omega \neq p_\omega\} \setminus \Xi$. Let $\tilde{p} \in \mathbb{R}^{\Omega_f}$ with $\tilde{p}_{\tilde{\omega}} = p'_{\tilde{\omega}}$ and $\tilde{p}_\omega = p'_\omega + \epsilon'_\omega$ for $\omega \neq \tilde{\omega}$. Note that by Lemma 2, we have $\Psi' \in D^f(\tilde{p})$. Thus, by monotonicity, we have $u^f(\Xi, p'+\epsilon') = u^f(\Xi, \tilde{p}) \leq u^f(\Psi', \tilde{p}) < u^f(\Psi', p'+\epsilon')$ contradicting the assumption that $\Xi \in D^f(p'+\epsilon')$.

Next we show that for each $\Xi \in D^f(p'+\epsilon')$ we have $\{\omega \notin \Psi' : p'_\omega \neq p_\omega\} \cap \Xi = \emptyset$. Suppose not, and there is a $\Xi \in D^f(p'+\epsilon')$ and a $\tilde{\omega} \in \{\omega \notin \Psi' : p'_\omega \neq p_\omega\} \cap \Xi$. Let $\tilde{p} \in \mathbb{R}^{\Omega_f}$ with $\tilde{p}_{\tilde{\omega}} = p'_{\tilde{\omega}}$ and $\tilde{p}_\omega = p'_\omega + \epsilon'_\omega$ for $\omega \neq \tilde{\omega}$. Note that by Lemma 2, we have $\Psi' \in D^f(\tilde{p})$. Thus, by monotonicity, we have $u^f(\Xi, p'+\epsilon') < u^f(\Xi, \tilde{p}) \leq u^f(\Psi', \tilde{p}) = u^f(\Psi', p'+\epsilon')$ contradicting the assumption that $\Xi \in D^f(p'+\epsilon')$. □

By Lemma 1, there exists another $\epsilon_1 > 0$ such that for every q with $\|q - (p'+\epsilon')\| < \epsilon_1$ we have $D^f(q) \subseteq D^f(p'+\epsilon')$. We may choose ϵ_1 such that $\epsilon_1 < \epsilon_0$.

Define prices $p(\epsilon), p'(\epsilon), \bar{p}(\epsilon), \underline{p}(\epsilon) \in \mathbb{R}^{\Omega_f}$ as follows:

$$p(\epsilon)_\omega := \begin{cases} p_\omega + \epsilon_1, & \text{if } \omega \in \Psi_{f\to}, \\ p_\omega - \epsilon_1, & \text{if } \omega \in \Omega_{f\to} \setminus \Psi, \\ p_\omega - \epsilon_1, & \text{if } \omega \in \Psi_{\to f}, \\ p_\omega + \epsilon_1, & \text{if } \omega \in \Omega_{\to f} \setminus \Psi. \end{cases}$$

$$p'(\epsilon)_\omega := \begin{cases} p'_\omega + \epsilon_0, & \text{if } \omega \in \Psi'_{f\to} \text{ and } p'_\omega \neq p_\omega, \\ p'_\omega - \epsilon_0, & \text{if } \omega \in \Omega_{f\to} \setminus \Psi' \text{ and } p'_\omega \neq p_\omega, \\ p'_\omega - \epsilon_0, & \text{if } \omega \in \Psi'_{\to f} \text{ and } p'_\omega \neq p_\omega, \\ p'_\omega + \epsilon_0, & \text{if } \omega \in \Omega_{\to f} \setminus \Psi' \text{ and } p'_\omega \neq p_\omega, \\ p_\omega + \epsilon_1, & \text{if } \omega \in \Psi_{f\to} \text{ and } p'_\omega = p_\omega, \\ p_\omega - \epsilon_1, & \text{if } \omega \in \Omega_{f\to} \setminus \Psi \text{ and } p'_\omega = p_\omega, \\ p_\omega - \epsilon_1, & \text{if } \omega \in \Psi_{\to f} \text{ and } p'_\omega = p_\omega, \\ p_\omega + \epsilon_1, & \text{if } \omega \in \Omega_{\to f} \setminus \Psi \text{ and } p'_\omega = p_\omega. \end{cases}$$
$$\bar{p}(\epsilon)_\omega := \max\{p(\epsilon)_\omega, p'(\epsilon)_\omega\}$$
$$\underline{p}(\epsilon)_\omega := \min\{p(\epsilon)_\omega, p'(\epsilon)_\omega\}.$$

By Lemma 2, we have $D^f(p(\epsilon)) = \{\Psi\}$. Moreover, we have $D^f(p'(\epsilon)) \subseteq D^f(p'+\epsilon') \subseteq D^f(p')$, $D^f(\bar{p}(\epsilon)) \subseteq D^f(\bar{p})$ and $D^f(\underline{p}(\epsilon)) \subseteq D^f(\underline{p})$.

Let $P := \{\tilde{p} \in \mathbb{R}^{\Omega_f} : \tilde{p}_\omega \in \{p(\epsilon)_\omega, p'(\epsilon)_\omega\} \text{ for all } \omega \in \Omega_f\}$. By Lemma 4, there is a single-valued selection $\tilde{D}^f : P \to 2^{\Omega_f}$ from D^f satisfying full substitutability and the laws of aggregate demand and supply. Let $\bar{\Psi} := D^f(\bar{p}(\epsilon))$ and $\underline{\Psi} := D^f(\underline{p}(\epsilon))$. As $D^f(p(\epsilon)) = \{\Psi\}$, we have $\tilde{D}^f(p(\epsilon)) = \Psi$. Moreover, by Claim 2 and as $\tilde{D}^f(p'(\epsilon)) \in D^f(p'+\epsilon')$, we have $\{\omega \in \Psi' : p'_\omega \neq p_\omega\} \subseteq \tilde{D}^f(p'(\epsilon))$ and $\{\omega \notin \Psi' : p'_\omega \neq p_\omega\} \cap \tilde{D}^f(p'(\epsilon)) = \emptyset$.

By FS of $\tilde{D}^f$ and since $\{\omega \in \Psi' : p'_\omega \neq p_\omega\} \subseteq \tilde{D}^f(p'(\epsilon))$, we have

$$\{\omega \in \Psi_{\to f} : p_\omega \geq p'_\omega\} \cup \{\omega \in \Psi'_{\to f} : p'_\omega > p_\omega\} \subseteq \bar{\Psi}_{\to f}.$$

Next we show that

$$\bar{\Psi}_{f\to} \subseteq \{\omega \in \Psi_{f\to} : p'_\omega \geq p_\omega\} \cup \{\omega \in \Psi'_{f\to} : p'_\omega > p_\omega\}.$$

Let $\bar{\omega} \in \bar{\Psi}_{f\to}$. We consider two cases. Either $\bar{p}_{\bar{\omega}} = p_{\bar{\omega}}$ or $\bar{p}_{\bar{\omega}} = p'_{\bar{\omega}} > p_{\bar{\omega}}$. In the first case, consider $\tilde{p} \in \mathbb{R}^{\Omega_f}$ with $\tilde{p}_\omega = \bar{p}(\epsilon)_\omega$ for $\omega \in \Omega_{\to f}$ and $\tilde{p}_\omega = p(\epsilon)_\omega$ for $\omega \in \Omega_{f\to}$. Let $\tilde{\Psi} := \tilde{D}^f(\tilde{p})$. By SSS of $\tilde{D}^f$, we have $\bar{\omega} \in \tilde{\Psi}_{f\to}$. By CSC, we have $\tilde{\Psi}_{f\to} \subseteq \Psi_{f\to}$ and hence $\bar{\omega} \in \Psi_{f\to}$.

Similarly, if $\bar{p}_{\bar{\omega}} = p'_{\bar{\omega}} > p_{\bar{\omega}}$, consider $\tilde{p} \in \mathbb{R}^{\Omega_f}$ with $\tilde{p}_\omega = \bar{p}(\epsilon)_\omega$ for $\omega \in \Omega_{\to f}$ and $\tilde{p}_\omega = p'(\epsilon)_\omega$ for $\omega \in \Omega_{f\to}$. Let $\tilde{\Psi} := \tilde{D}^f(\tilde{p})$. By SSS of $\tilde{D}^f$ we have $\bar{\omega} \in \tilde{\Psi}_{f\to}$. By CSC, we have $\tilde{\Psi}_{f\to} \subseteq (\tilde{D}^f(p'(\epsilon)))_{f\to}$. Since $\{\omega \notin \Psi' : p'_\omega \neq p_\omega\} \cap \tilde{D}^f(p'(\epsilon)) = \emptyset$ this implies $\bar{\omega} \in \Psi'_{f\to}$. A completely analogous proof shows that $\underline{\Psi}$ has the desired properties. Finally, by the laws of aggregate demand and supply for $\tilde{D}^f$, we have

$$|\underline{\Psi}_{\to f}| - |\underline{\Psi}_{f\to}| \geq |\Psi_{\to f}| - |\Psi_{f\to}| \geq |\bar{\Psi}_{\to f}| - |\bar{\Psi}_{f\to}|.$$

□

With this lemma we can prove the theorem.

Proof of Theorem 1. Let $\Xi \in \mathcal{E}(u,p)$ and $\Xi' \in \mathcal{E}(u,p')$. Define

$$\overline{\Xi} := \{\omega \in \Xi : p_\omega \geq p'_\omega\} \cup \{\omega \in \Xi' : p'_\omega > p_\omega\},$$
$$\underline{\Xi} := \{\omega \in \Xi : p'_\omega \geq p_\omega\} \cup \{\omega \in \Xi' : p_\omega > p'_\omega\}.$$

We show that $\overline{\Xi} \in \mathcal{E}(u,\bar{p})$ and $\underline{\Xi} \in \mathcal{E}(u,\underline{p})$. Let $f \in F$. By Lemma 5, with $\Psi = \Xi_f$ and $\Psi' = \Xi'_f$ there is a $\overline{\Psi}_f \in D^f(\bar{p})$ and a $\underline{\Psi}_f \in D^f(\underline{p})$ such that $\overline{\Xi}_{\to f} \subseteq \overline{\Psi}_{\to f}$, $\overline{\Psi}_{f\to} \subseteq \overline{\Xi}_{f\to}$, $\underline{\Psi}_{\to f} \subseteq \underline{\Xi}_{\to f}$ and $\underline{\Xi}_{f\to} \subseteq \underline{\Psi}_{f\to}$ and

$$|\underline{\Psi}_{\to f}| - |\underline{\Psi}_{f\to}| \geq |\Psi_{\to f}| - |\Psi_{f\to}| \geq |\bar{\Psi}_{\to f}| - |\bar{\Psi}_{f\to}|.$$

Note that this implies

$$|\underline{\Xi}_{\to f}| - |\underline{\Xi}_{f\to}| \geq |\underline{\Psi}_{\to f}| - |\underline{\Psi}_{f\to}| \geq |\Psi_{\to f}| - |\Psi_{f\to}| \geq |\bar{\Psi}_{\to f}| - |\bar{\Psi}_{f\to}| \geq |\bar{\Xi}_{\to f}| - |\bar{\Xi}_{f\to}|.$$

Summing the inequalities over all firms, we obtain

$$0 \geq \sum_{f\in F}(|\underline{\Psi}_{\to f}| - |\underline{\Psi}_{f\to}|) \geq 0 \geq \sum_{f\in F}(|\bar{\Psi}_{\to f}| - |\bar{\Psi}_{f\to}|) \geq 0.$$

Thus

$$|\underline{\Xi}| = \sum_{f\in F} |\underline{\Xi}_{f\to}| \leq \sum_{f\in F} |\underline{\Psi}_{f\to}| = \sum_{f\in F} |\underline{\Psi}_{\to f}| \leq \sum_{f\in F} |\underline{\Xi}_{\to f}| = |\underline{\Xi}|.$$

Therefore $\underline{\Xi}_f = \underline{\Psi}_f$ for each $f \in F$. Moreover,

$$|\overline{\Xi}| = \sum_{f\in F} |\overline{\Xi}_{\to f}| \leq \sum_{f\in F} |\overline{\Psi}_{\to f}| = \sum_{f\in F} |\overline{\Psi}_{f\to}| \leq \sum_{f\in F} |\overline{\Xi}_{\to f}| = |\overline{\Xi}|.$$

Therefore $\overline{\Xi}_f = \overline{\Psi}_f$ for each $f \in F$.

Next we show that the above construction implies the rural hospital theorem: Note that for each $f \in F$ we have

$$|\underline{\Xi}_{\to f}| - |\underline{\Xi}_{f\to}| \geq |\Xi_{\to f}| - |\Xi_{f\to}| \geq |\bar{\Xi}_{\to f}| - |\bar{\Xi}_{f\to}|.$$

Summing the inequalities over all f, we obtain

$$0 = \sum_{f\in F} |\underline{\Xi}_{\to f}| - \sum_{f\in F} |\underline{\Xi}_{f\to}| \geq \sum_{f\in F} |\Xi_{\to f}| - \sum_{f\in F} |\Xi_{f\to}| \geq \sum_{f\in F} |\bar{\Xi}_{\to f}| - \sum_{f\in F} |\bar{\Xi}_{f\to}| = 0.$$

Thus, for each $f \in F$ we have

$$|\underline{\Xi}_{\to f}| - |\underline{\Xi}_{f\to}| = |\Xi_{\to f}| - |\Xi_{f\to}| = |\bar{\Xi}_{\to f}| - |\bar{\Xi}_{f\to}|.$$

Now observe that by the above reasoning (with $[\bar{\Xi}, \bar{p}]$ in the role of $[\Xi, p]$) for the set of trades

$$\Xi'' := \{\omega \in \bar{\Xi} : \bar{p}_\omega = p_\omega\} \cup \{\omega \in \Xi' : \bar{p}_\omega > p'_\omega\}.$$

we have $\Xi'' \in \mathcal{E}(u, p')$ and for each $f \in F$ we have

$$|\Xi''_{\to f}| - |\Xi''_{f\to}| = |\bar{\Xi}_{\to f}| - |\bar{\Xi}_{f\to}|.$$

Since

$$|\bar{\Xi}_{\to f}| - |\bar{\Xi}_{f\to}| = |\Xi_{\to f}| - |\Xi_{f\to}|,$$

this concludes the proof. □

E Proof of Theorem 3

Proof. We prove the statement for terminal sellers. The proof for terminal buyers is very similar. Let $F' \subseteq F$ be the set of terminal sellers. Let $\mathcal{U} = \times_{f\in F}\mathcal{U}_f$ where for $f \in F'$ the set $\mathcal{U}_f$ is the set of unit supply and BWP utility functions and for each $f \in F \setminus F'$ the set $\mathcal{U}_f$ is the set of BWP, FS, LAD and LAD utility functions. Let $\mathcal{M} : \mathcal{U} \to \mathcal{A}$ be a seller optimal mechanism.

First we establish that $\mathcal{M}$ is immune to truncation strategies.

Claim 3. *Let $f \in F'$. Let $u, \tilde{u} \in \mathcal{U}$ with $\tilde{u}^{-f} = u^{-f}$ and let $[\Psi, p]$ be a seller optimal equilibrium under u. If $\Psi_f \neq \emptyset$, $\tilde{u}^f(\omega, \cdot) = u^f(\omega, \cdot)$ for each $\omega \in \Omega_{f\to}$ and $\tilde{u}^f(\emptyset) > \tilde{u}^f(\Psi, p)$, then for each equilibrium $[\tilde{\Psi}, \tilde{p}]$ under $\tilde{u}$, we have $\tilde{\Psi}_f = \emptyset$.*

Proof. Suppose not. Then $\tilde{\Psi}_f \neq \emptyset$. Let $\tilde{\Psi}_f = \{\tilde{\omega}\}$. Note that also $\{\tilde{\omega}\} \in D^f(\tilde{p})$. Thus $\tilde{\Psi} \in \mathcal{E}(u, \tilde{p})$. But since $u^f(\tilde{\omega}, \tilde{p}) = \tilde{u}^f(\tilde{\omega}, \tilde{p}) \geq \tilde{u}^f(\emptyset) > \tilde{u}^f(\tilde{\omega}, p_{\tilde{\omega}}) = u^f(\tilde{\omega}, p_{\tilde{\omega}})$ this contradicts the seller optimality of $[\Psi, p]$. □

Second we establish that $\mathcal{M}$ is immune to certain strategies where a single terminal seller changes the utility function for one trade.

Claim 4. *Let $f \in F'$. Let $u, \hat{u} \in \mathcal{U}$ with $\hat{u}^{-f} = u^{-f}$ and such that there is a $\hat{\omega} \in \Omega_{f\to}$ with $\hat{u}^f(\omega, \cdot) = u^f(\omega, \cdot)$ for $\omega \neq \hat{\omega}$ and $\hat{u}^f(\emptyset) = u^f(\emptyset)$. Let $[\bar{\Psi}, \bar{p}]$ be a seller-optimal equilibrium under u. If $\bar{\Psi}_f = \{\hat{\omega}\}$ and $\hat{u}^f(\hat{\omega}, \bar{p}_{\hat{\omega}}) = u^f(\hat{\omega}, \bar{p}_{\hat{\omega}})$, then $[\bar{\Psi}, \bar{p}]$ is a seller optimal equilibrium under $\hat{u}$.*

Proof. Let $[\hat{\Psi}, \hat{p}]$ be a seller optimal equilibrium under $\hat{u}$. Note that $D^f(\bar{p}) = \hat{D}^f(\bar{p})$. Thus $[\bar{\Psi}, \bar{p}]$ is an equilibrium under $\hat{u}$.

First we show that $\hat{p}_{\hat{\omega}} = \bar{p}_{\hat{\omega}}$. Suppose not. Then $\hat{p}_{\hat{\omega}} > \bar{p}_{\hat{\omega}}$ and in particular $u^f(\hat{\Psi}, \hat{p}) \geq u^f(\hat{\omega}, \hat{p}_{\hat{\omega}}) > u^f(\hat{\omega}, \bar{p}_{\hat{\omega}}) = u^f(\bar{\Psi}, \bar{p})$. There are two cases. Either $\hat{\Psi}_f = \{\hat{\omega}\}$, or there is a $\tilde{\omega} \neq \hat{\omega}$ with $\hat{\Psi}_f = \{\tilde{\omega}\}$. In the first case, consider the utility function $\tilde{u}^f$ obtained from u^f by truncating as follows: $\tilde{u}^f(\omega, \cdot) = u^f(\omega, \cdot)$ for all $\omega \in \Omega_{f\to}$ and $u^f(\hat{\omega}, \bar{p}_{\hat{\omega}}) < \tilde{u}^f(\emptyset) < u^f(\hat{\omega}, \hat{p}_{\hat{\omega}})$. By Claim 3, for each equilibrium $[\Psi, p]$ under $\tilde{u}$we have $\Psi_f = \emptyset$. Define the utility function $\tilde{u}^f_*$ by $\tilde{u}^f_*(\hat{\omega}, \cdot) = \tilde{u}^f(\hat{\omega}, \cdot) = u(\hat{\omega}, \cdot)$, by $\tilde{u}^f_*(\omega, \cdot) = -\infty$ for each $\omega \neq \hat{\omega}$, and $\tilde{u}^f_*(\emptyset) = \tilde{u}^f(\emptyset)$. As for each equilibrium $[\Psi, p]$ under $\tilde{u}$we have $\Psi_f = \emptyset$, we have $\mathcal{E}(\tilde{u}) \subseteq \mathcal{E}(\tilde{u}_*)$, and in particular, there is an equilibrium $[\tilde{\Psi}, \tilde{p}]$ under $\tilde{u}_*$ with $\tilde{\Psi}_f = \emptyset$. Observe however that $\tilde{u}^f_*(\hat{\omega}, \hat{p}_{\hat{\omega}}) = \tilde{u}^f(\hat{\omega}, \hat{p}_{\hat{\omega}}) = u(\hat{\omega}, \hat{p}_{\hat{\omega}}) > \tilde{u}^f_*(\emptyset)$. Thus $\tilde{D}^f_*(\hat{p}) = \{\{\hat{\omega}\}\}$ and $[\hat{\Psi}, \hat{p}]$ is an equilibrium under $\tilde{u}_*$ with $\tilde{u}^f_*(\hat{\Psi}, \hat{p}) > \tilde{u}^f_*(\emptyset)$. This contradicts the rural hospitals theorem (the second part of Theorem 1).

In the second case, consider the utility function $\tilde{u}^f$ obtained from u^f by truncating as follows: $\tilde{u}^f(\omega, \cdot) = u^f(\omega, \cdot)$ for all $\omega \in \Omega_{f\to}$ and $u^f(\bar{\Psi}, \bar{p}) < \tilde{u}^f(\emptyset) < u^f(\tilde{\omega}, \hat{p}_{\tilde{\omega}})$. By Claim 3, for each equilibrium $[\Psi, p]$ under $\tilde{u}$we have $\Psi_f = \emptyset$. Define the utility function $\tilde{u}^f_*$ by $\tilde{u}^f_*(\tilde{\omega}, \cdot) = \tilde{u}^f(\tilde{\omega}, \cdot) = \hat{u}(\tilde{\omega}, \cdot) = u(\tilde{\omega}, \cdot)$, by $\tilde{u}^f_*(\omega, \cdot) = -\infty$ for each $\omega \neq \tilde{\omega}$ and $\tilde{u}^f_*(\emptyset) = \tilde{u}^f(\emptyset)$. As for each equilibrium $[\Psi, p]$ under $\tilde{u}$we have $\Psi_f = \emptyset$, we have $\mathcal{E}(\tilde{u}) \subseteq \mathcal{E}(\tilde{u}_*)$, and in particular, there is an equilibrium $[\tilde{\Psi}, \tilde{p}]$ under $\tilde{u}_*$ with $\tilde{\Psi}_f = \emptyset$. Observe however that $\tilde{u}^f_*(\tilde{\omega}, \hat{p}_{\tilde{\omega}}) = \tilde{u}^f(\tilde{\omega}, \hat{p}_{\tilde{\omega}}) = u(\tilde{\omega}, \hat{p}_{\tilde{\omega}}) > \tilde{u}^f_*(\emptyset)$. Thus $\tilde{D}^f_*(\hat{p}) = \{\{\tilde{\omega}\}\}$ and $[\hat{\Psi}, \hat{p}]$ is an equilibrium under $\tilde{u}_*$ with $\tilde{u}^f_*(\hat{\Psi}, \hat{p}) > \tilde{u}^f_*(\emptyset)$. This contradicts the rural hospitals theorem (the second part of Theorem 1).

We have established that $\hat{p}_{\hat{\omega}} = \bar{p}_{\hat{\omega}}$. But then for each $f' \in F$, we have $\hat{\Psi}_{f'} \in D^{f'}(\hat{p})$ and $(\hat{\Psi}, \hat{p})$ is an equilibrium allocation under u as well. For each $f' \in F'$ we have $\hat{u}^{f'}(\hat{\Psi}, \hat{p}) = u^{f'}(\hat{\Psi}, \hat{p}) \leq u^{f'}(\bar{\Psi}, \bar{p}) = u^{f'}(\bar{\Psi}, \bar{p})$ and as $(\hat{\Psi}, \hat{p})$ is seller optimal under $\hat{u}$, we have $\hat{u}^{f'}(\hat{\Psi}, \hat{p}) = u^{f'}(\bar{\Psi}, \bar{p})$ for each $f' \in F'$. Thus $(\bar{\Psi}, \bar{p})$ is a seller optimal equilibrium allocation under $\hat{u}$. □

With the lemma, we can prove the result. Suppose there are profiles $u, \tilde{u} \in \mathcal{U}$ such that $\tilde{u}^{-F'} = u^{-F'}$ and for each $f \in F'$, we have

$$u^f(\mathcal{M}(\tilde{u})) > u^f(\mathcal{M}(u)),$$

Let $\mathcal{M}(u) = (\bar{\Psi}, \bar{p})$ and $\mathcal{M}(\tilde{u}) = (\tilde{\Psi}, \tilde{p})$. By BWP for u and for $\tilde{u}$, there exists a $K > 0$ such that for all $p \in \mathbb{R}^\Omega$ and all $\omega \in \Omega$, if $\omega \in \Psi \in D^{b(\omega)}(p)$ then $p_\omega < K$, if $\omega \in \Psi \in D^{s(\omega)}(p)$ then $p_\omega > -K$, if $\omega \in \Psi \in \tilde{D}^{b(\omega)}(p)$ then $p_\omega < K$, if $\omega \in \Psi \in \tilde{D}^{s(\omega)}(p)$. Thus we can specify prices for non realized trades at (Ψ, p) to obtain $p \in \mathcal{E}(u)$ with $p \in [-K, K]^\Omega$ and similarly we can specify prices for non realized trades at $(\tilde{\Psi}, \tilde{p})$ to obtain $\tilde{p} \in \mathcal{E}(\tilde{u})$ with $\tilde{p} \in [-K, K]^\Omega$.

Now we define for each $f \in F'$, a $\hat{u}^f \in \mathcal{U}_f$ as follows: Note that $\tilde{\Psi}_f \neq \emptyset$ as $u^f(\tilde{\Psi}, \tilde{p}) > u^f(\bar{\Psi}, \bar{p}) \geq u^f(\emptyset)$. Let $\tilde{\omega} \in \tilde{\Psi}$ be the unique trade in $\tilde{\Psi}$ such that $s(\tilde{\omega}) = f$. We let $\hat{u}^f(\omega, \cdot) = u^f(\omega, \cdot)$ for $\omega \neq \tilde{\omega}$ and we let $\hat{u}^f(\emptyset) = u^f(\emptyset)$. To construct $\hat{u}^f(\tilde{\omega}, \cdot)$ we proceed as follows: Define $\bar{u} := \max_{\omega \in \Omega_{f\to}} u^f(\omega, K)$ and $\hat{u}^f(\tilde{\omega}, \tilde{p}_{\tilde{\omega}}) = \max\{\bar{u}, \tilde{u}^f(\tilde{\omega}, \tilde{p}_{\tilde{\omega}})\}$. Define $\hat{u}^f(\tilde{\omega}, \bar{p}_{\tilde{\omega}}) := u^f(\tilde{\omega}, \bar{p}_{\tilde{\omega}})$. Note that $\tilde{p}_{\tilde{\omega}} > \bar{p}_{\tilde{\omega}}$ and by construction $\hat{u}^f(\tilde{\omega}, \tilde{p}_{\tilde{\omega}}) \geq \bar{u} \geq u^f(\tilde{\omega}, K) > u^f(\tilde{\omega}, \bar{p}_{\tilde{\omega}}) = \hat{u}^f(\tilde{\omega}, \bar{p}_{\tilde{\omega}})$. We can choose any continuous and monotonic extension for prices other than $\bar{p}_{\tilde{\omega}}$ and $\tilde{p}_{\tilde{\omega}}$.

Proceeding in this way for each $f \in F'$, we have constructed $\hat{u}$ such that $\hat{u}^{-F'} = u^{-F'}$, and for each $f \in F'$ we have $\hat{u}^f(\tilde{\omega}, \bar{p}_{\tilde{\omega}}) = u^f(\tilde{\omega}, \bar{p}_{\tilde{\omega}})$, $\hat{u}^f(\omega, \cdot) = u^f(\omega, \cdot)$ for $\omega \neq \tilde{\omega}$ and $\tilde{u}^f(\emptyset) = u^f(\emptyset)$. Extend $(\tilde{\Psi}, \tilde{p})$ to an arrangement $[\tilde{\Psi}, \tilde{p}]$ as follows: For each trade $\omega \in \Omega$ if there a $\tilde{p}_\omega \in \mathbb{R}$ such that $u^{b(\omega)}$ is an equilibrium under $\hat{u}$, and for each seller-optimal equilibrium $[\hat{\Psi}, \hat{p}]$ under $\hat{u}$ we have $\hat{\Psi}_f = \tilde{\Psi}_f$ for each $f \in F'$. Thus for each $f \in F'$ we have $u^f(\hat{\Psi}, \hat{p}) \geq u^f(\tilde{\Psi}, \tilde{p})$.

Let $f \in F'$. By Claim 4, $[\bar{\Psi}, \bar{p}]$ is a seller optimal equilibrium for $(\hat{u}^f, u^{-f})$. Iterating for all $f \in F'$, $[\bar{\Psi}, \bar{p}]$ is a seller optimal equilibrium under $\hat{u}$. We have derived a contradiction. □